ON THE THEORY OF EMISSION OF ALPHA-PARTICLES FROM RADIOACTIVE NUCLEI*

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ABSTRACT. The rectangular potential hole of Condon-Gurney-Gamow model has been replaced by an exponential potential function $V = -V_0 e^{-r/r_0}$ and an expression has been worked out for the decay constant λ. It is found that the exact form and magnitude of the potential function inside the nucleus plays a very insignificant rôle in the problem. In the two limiting cases V_0 very small and V_0 very large, the decay constants λ_0 and λ_∞ are both independent of V_0 and ratio λ_∞/λ_0 is of the order 5 only. The nuclear radius r_0 has been calculated using the experimental data for the decay constant λ and the energy E and all the radii except those of ThC', RaC' are found to lie between 7.8 and 9.7 × 10^{-13} cm. which is of the right order of magnitude and agrees in a satisfactory manner with those obtained by the previous workers with the one-body picture of the nucleus.

A satisfactory explanation of the emission of alpha-particles from certain radioactive substances was first given by Condon and Gurney (1929) and independently, by Gamow (1928), who considered the problem as a transmission of alpha particles through the nuclear potential barrier. For ease in calculation the nucleus was represented by a rectangular potential hole of a certain constant depth and a width r_0 where r_0 is defined as the nuclear radius. In the range $r_0 \leq r \geq \infty$ the potential function was taken to be the Coulomb one, *i.e.*, $V = \frac{2Ze^2}{r}$ where Z is the atomic number of the product nucleus and $2e$ is the charge of the alpha-particle. Such a picture of the nucleus is very crude. As a matter of fact, experimental evidences go to prove definitely that the nucleus consists of protons and neutrons held together by strong and short-range exchange forces. Thus the nuclear problems must be regarded as many-body problems and not as one-body problems. The extreme ease, with which energy is interchanged between densely packed particles inside the nucleus, can be easily understood with this model. The emission of an alpha-particle into free space after crossing the potential barrier which surrounds the nucleus is understood in this way, *viz.*, the interaction between the several constituents of the nucleus leaves an alpha-particle with a certain kinetic energy; possessed of this kinetic energy the alpha-particle penetrates through the barrier. The ideal calculation would be to find out the probability of the formation of alpha-particles from neutrons and protons

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and their being left with a certain kinetic energy and the next task would be to find out the penetrability of the alpha-particles through the nuclear potential barrier. This is how the problem should be tackled if the nucleus is considered as a many-body problem. Unfortunately the mathematical complexities are so great that the many-body problem may remain unsolved for a considerably long time.

The exact law of force between an alpha-particle and other nuclear particles is at present unknown. It may be expected that the exact form of the potential function inside the nucleus will not drastically modify the results obtained in the rectangular potential hole model but a detailed mathematical analysis of this expected result has not been attempted. Bethe (1937) mentions that the exact value of the potential depth is of no great importance. It was therefore thought necessary to examine this problem in detail.

In the present paper, the rectangular potential hole of Condon-Gurney-Gamow model has been replaced by an exponential function given by,

$$V = -V_0 e^{-r/r_0} \qquad \dots \quad (1)$$

and an expression has been worked out for the decay constant λ. It has been found that the decay constant λ does not sensibly depend on the magnitude of V_0 and in the two limiting cases, V_0 very small and V_0 very large it is in fact independent of V_0 and the ratio λ_∞/λ_0 is of the order 5 only. It is entirely a new result and provides a clear and detailed analysis and in support of a hitherto expected result. Using the experimental data for the decay constant, the nuclear radii have been calculated and the values obtained are quite satisfactory and agree well with those given by previous workers with the one-body picture of the nucleus. In view of the more correct many-body picture of the nucleus, the results presented in this paper are purely of theoretical interest.

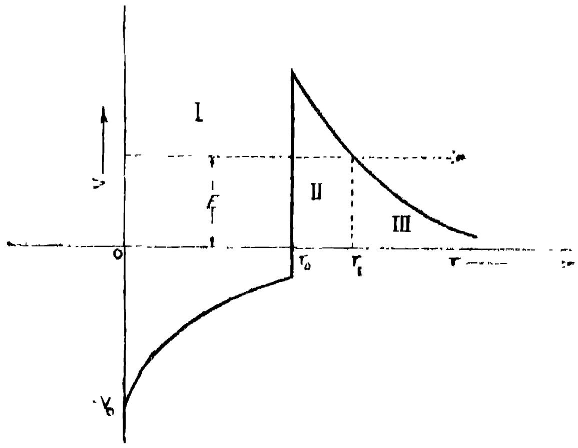

FIG. 1

The exponential potential function is shown in Fig. 1 as a function of r the distance from the centre of the nucleus. If we consider alpha-particles of a given energy E, we may divide the whole space into three regions as indicated in Fig. 1.

(*i*) The interior of the nucleus $0 < r \leq r_0$, where the potential energy

$$V = -V_0 e^{-r/r_0}$$

(*ii*) The region of the potential barrier $r_0 \leq r \leq r_1$ where r_1 is the classical distance of closest approach of an alpha-particle of energy E falling on the nucleus from outside. In this region the potential energy is greater than the energy of the alpha-particle.

(*iii*) The outside region $r > r_1$ in which the potential energy is less than E.

We consider the case when the wave function of the alpha-particle is spherically symmetrical. This corresponds to an alpha-particle of orbital momentum $l=0$ or what is known as the *s*-state of the particle. The wave function of an alpha-particle can then be written in the form,

$$\phi = \frac{\psi}{r}$$

where ψ satisfies the equation

$$\frac{d^2\psi}{dr^2} + \frac{2M}{\mathbf{h}^2}(E-V)\psi = 0$$

where M is the mass of an alpha-particle and **h** is Planck's constant divided 2π. We take the values (Birge, 1941)

$$M = 6.644 \times 10^{-24} \text{ gm.}$$

$$\mathbf{h} = 1.054 \times 10^{-27} \text{ erg-sec}$$

In the three regions marked in the figure 1, the equations are

Region I, $$\frac{d^2\psi}{dr^2} + \frac{2M}{\mathbf{h}^2}\left(E + V_0 e^{-r/r_0}\right)\psi = 0 \qquad \ldots (2)$$

Region II, $$\frac{d^2\psi}{dr^2} + \frac{2M}{\mathbf{h}^2}(E-V)\psi = 0 \qquad \ldots (3)$$

Region III, $$\frac{d^2\psi}{dr^2} + \frac{2M}{\mathbf{h}^2}(E-V)\psi = 0 \qquad \ldots (4)$$

In equation (3), $V = \frac{2Ze^2}{r}$ and $E-V$ is negative whereas in equation (4) V is again $\frac{2Ze^2}{r}$ but $E-V$ is positive.

First of all we give the solution of equation (2).

Putting

$$y = e^{-r/2r_0}$$

we have $$\frac{d\psi}{dr} = -\frac{y}{2r_0}\cdot\frac{d\psi}{dy} \quad \text{and} \quad \frac{d^2\psi}{dr^2} = \left(\frac{y}{2r_0}\right)^2\left\{\frac{d^2\psi}{dy^2} + \frac{1}{y}\frac{d\psi}{dy}\right\}$$

Hence the equation (2) transforms itself to

$$\frac{d^2\psi}{dy^2}+\frac{1}{y}\frac{d\psi}{dy}+\frac{2M}{\mathbf{h}^2}(2r_0)^2\left(V_0+\frac{E}{y^2}\right)\psi=0 \quad \ldots \quad (5)$$

We define $$p^2=\frac{2M}{\mathbf{h}^2}.(2r_0)^2.V_0 \quad \ldots \quad (6)$$

$$\mu^2=-\frac{2M}{\mathbf{h}^2}.(2r_0)^2.E=-p^2\frac{E}{V_0}=i^2n^2. \quad \ldots \quad (7)$$

Therefore the equation (5) takes the form,

$$\frac{d^2\psi}{dy^2}+\frac{1}{y}\frac{d\psi}{dy}+\left(p^2-\frac{\mu^2}{y^2}\right)\psi=0 \quad \ldots \quad (8)$$

This is a standard type of equation and is known as Bessel's equation and has been well-studied by mathematicians. Here μ is not an integer, therefore the general solution of this equation is,

$$\psi_I=C.J_{\mu}(py)+D.J_{-\mu}(py)$$

or $$\psi_I=C.J_{ni}(pe^{-r/2r_0})+D.J_{-ni}(pe^{-r/2r_0}) \quad \ldots \quad (9)$$

where C and D are constants to be determined by the boundary conditions. The first term corresponds to a wave travelling towards the right and the second one to a wave travelling towards the left after being reflected at the potential barrier.

We next consider the equation (3) in which case $E-V$ is negative. Let us define

$$\phi(r)=\frac{2M}{\mathbf{h}^2}\left(\frac{2Ze^2}{r}-E\right), \text{ (positive)} \quad \ldots \quad (10)$$

The solution ψ_{II} in this region is of an exponential character rather than of a wave type. As has been shown by Jeffreys (1924) and later on, independently by Kramers (1926), the two fundamental solutions of (3) approximate very closely to

$$\psi_1=B_1\left\{\phi(r)\right\}^{-\frac{1}{4}}.\exp.\left\{\int_r^{r_1}\left[\phi(r)\right]^{\frac{1}{2}}dr\right\}$$

$$\psi_2=\tfrac{1}{2}B_2.\left\{\phi(r)\right\}^{-\frac{1}{4}}.\exp.\left\{-\int_r^{r_1}\left[\phi(r)\right]^{\frac{1}{2}}dr\right\} \quad \ldots \quad (11)$$

It is to be noted that ψ_1 decreases from the nucleus outwards and ψ_2 increases. At the boundary of the nucleus, function $\psi_1(r_0)$ will be much greater than $\psi_2(r_0)$ because ψ_1 contains an exponential with a large positive exponent while ψ_2 contains one with a large negative exponent.

Lastly we consider the equation (4) in which case $E-V$ is positive. At large distances from the nucleus, V is very small and therefore it can be neglected. The equation then becomes

$$\frac{d^2\psi}{dr^2} + \frac{2M}{\mathbf{h}^2} E\psi = 0$$

of which a general solution is

$$\psi = Ae^{ikr} + Be^{-ikr}$$

where $$k^2 = \frac{2M}{\mathbf{h}^2} E$$

and A and B are some constants. By multiplying it with the time factor $\exp.\left(\frac{-iEt}{\mathbf{h}}\right)$, it can be seen that the first term represents an outgoing wave while the second term gives an incoming wave. In the physical problem considered here, the alpha-particle leaves the nucleus and none comes towards it from outside, hence we put $B = 0$.

Therefore, $$\psi = Ae^{ikr} \qquad \dots \quad (12)$$

To obtain the wave function for smaller values of r, it is again convenient to use the Wenzel- Kramers- Brillouin (1932) approximation in the form given by Kramers (1926). According to this method, the two fairly good approximate solutions of the equation (4), which fit in smoothly at $r = r_1$ with the two solutions given in (11) respectively are,

$$\psi_3 = \{\phi(r)\}^{-\frac{1}{4}} \cos\left\{\int_{r_1}^{r} |\phi(r)|^{\frac{1}{2}} dr + \frac{\pi}{4}\right\}$$
$$\psi_4 = \{\phi(r)\}^{-\frac{1}{4}} \cos\left\{\int_{r_1}^{r} [\phi(r)]^{\frac{1}{2}} dr - \frac{\pi}{4}\right\} \qquad \dots \quad (13)$$

The most general solution of the equation (4) is then,

$$\psi_{III} = B_1\psi_3 + B_2\psi_4 \qquad \dots \quad (14)$$

In order that (14) may be of the same form as (12), we must choose

$$B_2 = iB_1$$

Since, for large r, V can be neglected $\phi(r) = k^2$ and therefore,

$$\psi_{III} \longrightarrow B_1 k^{-\frac{1}{2}} e^{ikr} \qquad \dots \quad (15)$$

Comparing (12) and (15) we have,

$$A = B_1 k^{-\frac{1}{2}} \qquad \dots \quad (16)$$

At the boundary of the nucleus $r = r_0$, the wave function of the region I will be equal to that of region II and so will their first derivatives be. As has already been pointed, $\psi_1(r_0)$ will be much greater than $\psi_2(r_0)$, hence we may neglect $\psi_2(r_0)$ and put,

$$\psi_I(r_0) = \psi_1(r_0); \left(\frac{d\psi_I}{dr}\right)_{r_0} = \left(\frac{d\psi_1}{dr}\right)_{r_0}$$

or $$^{*}CJ_{n i}(pe^{-\frac{1}{2}})+DJ_{-n i}(pe^{-\frac{1}{2}})=B_1\{\phi(r_0)\}^{-\frac{1}{4}}\ \exp.\left\{\int_{r_0}^{r_r}[\phi(r)]^{\frac{1}{2}}dr\right\}$$

and, $$CJ'_{n i}(pe^{-\frac{1}{2}})+DJ'_{-n i}(pe^{-\frac{1}{2}})=\frac{2r_0}{pe^{-\frac{1}{2}}}.B_1.\{\phi(r_0)\}^{\frac{1}{4}}\ \exp.\left\{\int_{r_0}^{r_r}[\phi(r_0)]^{\frac{1}{2}}dr\right\}.$$

where $J'_{\mu}(pe^{-\frac{1}{2}})=\left\{\frac{\partial J_{\mu}(pe^{-r/2r_0})}{\partial(pe^{-r/2r_0})}\right\}_{r=r_0}$ and we have neglected a term $\frac{1}{2}\phi^{-\frac{3}{4}}.\frac{d\phi}{dr}$ in comparison with $\phi^{\frac{1}{4}}$; as it is numerically very small.

We define,

$$\left.\begin{aligned} pe^{-\frac{1}{2}}&=x\\ J_{n i}(x)&=a\\ J'_{n i}(x)&=a'\\ J_{-n i}(x)&=\beta\\ J'_{-n i}(x)&=\beta'\\ \frac{2r_0}{x}&=m\\ \int_{r_0}^{r_r}[\phi(r)]^{\frac{1}{2}}dr&=P \end{aligned}\right\} \qquad \ldots (17)$$

Therefore,

$$Ca+D\beta-B_1\{\phi(r_0)\}^{\frac{1}{4}}e^{P}=0$$

$$Ca'+D\beta'-B_1m\{\phi(r_0)\}^{\frac{1}{4}}e^{P}=0.$$

This gives us

$$\frac{C}{\beta'\{\phi(r_0)\}^{-\frac{1}{4}}e^{P}-m\beta\{\phi(r_0)\}^{\frac{1}{4}}e^{P}}=\frac{D}{ma\{\phi(r_0)\}^{\frac{1}{4}}e^{P}-a'\{\phi(r_0)\}^{-\frac{1}{4}}e^{P}}=\frac{B_1}{a\beta'-a'\beta}.$$

or $$B_1=C.\frac{a\beta'-a'\beta.}{\beta'\{\phi(r_0)\}^{-\frac{1}{4}}-m\beta\{\phi(r_0)\}^{\frac{1}{4}}}.e^{-P} \qquad \ldots (18)$$

The decay constant λ is given by

$$dN=-\lambda N dt$$

or $$\lambda=-\frac{1}{N}.\frac{dN}{dt}.$$

where $\frac{1}{N}\frac{dN}{dt}$ is the ratio of all the particles coming out per second from the nucleus to the total number incident. Thus we have,

$$\lambda=4\pi r^2.v.|\Phi|^2$$

$$=4\pi.v.|A|^2.$$

$$=\frac{4\pi v}{k}.|B_1|^2$$

* $e^{-\frac{1}{2}}$ is $\exp.(-\frac{1}{2})$ and should not be confused with the electronic charge, for which the greek symbol ϵ (epsilon) has been used.

Now $\dfrac{v}{k} = \dfrac{\mathbf{h}}{M}$ and $\phi(r_0) = \dfrac{2M}{\mathbf{h}^2}.(V_1 - E)$ where $V_1 = \dfrac{2Ze^2}{r_0}$ and defining

$$\gamma = \alpha\beta' - \alpha'\beta \qquad \dots \quad (19)$$

we have,

$$\lambda = \frac{4\pi\mathbf{h}}{M} . |C|^2 . \left| \frac{\gamma}{m\{\phi(r_0)\}^{\frac{1}{2}}\beta\left[\dfrac{\beta'}{\beta m\{\phi(r_0)\}^{\frac{1}{2}}} - 1\right]}\right|^2 . e^{-2P} \qquad \dots \quad (20)$$

It has been proved on page 43, Theory of Bessel Functions, Watson (1922), that if ν be not an integer,

$$J_\nu(z)J'_{-\nu}(z) - J_{-\nu}(z)J'_\nu(z) = -\frac{2\sin\nu\pi}{\pi z}$$

Replacing ν by in and the argument z by x, this becomes identical with γ defined in (19). Therefore we have,

$$\gamma = -\frac{2\sin(in\pi)}{\pi x} \quad \text{and} \quad \gamma^2 = \frac{4\sinh^2 n\pi}{\pi^2 x^2} \qquad \dots \quad (21)$$

Using the expression for m and $\phi(r_0)$, we have

$$m\{\phi(r_0)\}^{\frac{1}{2}} = \frac{2r_0}{x} . \frac{\sqrt{2M}}{\mathbf{h}} . (V_1 - E)^{\frac{1}{2}} = e^{\frac{1}{2}}.\left(\frac{V_1 - E}{V_0}\right)^{\frac{1}{2}} \qquad \dots \quad (22)$$

Also, $$m^2x^2 = (2r_0)^2$$

Hence,

$$\lambda = -\frac{4\mathbf{h}^2}{\sqrt{2}.\pi M^{\frac{3}{2}} r_0^2} . \frac{\sinh^2 n\pi}{(V_1 - E)^{\frac{1}{2}}} . \left|\frac{C}{\beta}\right|^2 \frac{e^{-2P}}{\left|\dfrac{\beta'}{\beta e^{\frac{1}{2}}}\left(\dfrac{V_0}{V_1 - E}\right)^{\frac{1}{2}} - 1\right|^2} \qquad \dots \quad (23)$$

$|C|^2$ can be determined from the condition that the intensity of the incident wave is unity. Therefore we have,

$$4\pi|C|^2 \int_0^{r_0} J_{in}(pe^{-r/2r_0}) . J_{-in}(pe^{-r/2r_0}) . dr = 1$$

Putting $z = pe^{-r/2r_0}$, $dr = -\dfrac{2r_0}{z}dz$ then $-8\pi r_0|C|^2.\displaystyle\int_p^x \frac{J_{in}(z)J_{-in}(z)}{z}dz = 1$

To perform this integration, we make use of the relation (7), page 147, Watson (1922), *viz.*,

$$J_\nu(z)J_{-\nu}(z) = \sum_{s=0}^{\infty} \frac{(-1)^s(\frac{1}{2}-z)^{2s}.(2s)!}{(s!)^2\Gamma(\nu+s+1)\Gamma(-\nu+s+1)}$$

Putting $$\nu = in$$

$$\frac{J_{in}(z)J_{-in}(z)}{z} = \sum_{s=0}^{\infty} \frac{(-1)^s.(2s)!\,z^{2s-1}}{2^{2s}.(s!)^2\Gamma(in+s+1)\Gamma(-in+s+1)}$$

Therefore we have,

$$\int_p^x \frac{J_{in}(z)J_{-in}(z)}{z}\,dz = \frac{1}{\Gamma(1+in)\Gamma(1-in)}\log\frac{x}{p} +$$

$$\sum_{s=1}^{\infty}\frac{(-1)^s(2s-1)!(x^{2s}-p^{2s})}{2^{2s}.(s!)^2\Gamma(in+s+1)\Gamma(-in+s+1)}$$

$$= -\frac{1}{2\Gamma(1-in)\Gamma(1+in)} + \sum_{s=1}^{\infty}\frac{(-1)^{s+1}(2s-1)!\,.\,p^{2s}(1-e^{-s})}{2^{2s}.(s!)^2\Gamma(in+s+1)\Gamma(-in+s+1)} \qquad (24)$$

For small values of p and hence of V_0, we can neglect all the terms containing p^2 and higher powers and we have,

$$|C|^2 = \frac{1}{4\pi r_0}\,.\,\Gamma(1+in)\Gamma(1-in) = \frac{1}{4\pi r_0}\,.\,\frac{n\pi}{\sinh n\pi} = \frac{n}{4r_0\sinh n\pi} \qquad \ldots (25)$$

If p and hence V_0 be not negligible, we have to retain three or four terms of the series containing p. The the range of integration being small, a sufficiently accurate value is given by

$$\int_p^x \frac{J_{in}(z)J_{-in}(z)}{z}\,dz = J_{in}\left(\frac{x+p}{2}\right).J_{-in}\left(\frac{x+p}{2}\right).\log\frac{x}{p} = -\tfrac{1}{2}J_{in}\left(\frac{x+p}{2}\right)J_{-in}\left(\frac{x+p}{2}\right)$$

or

$$|C|^2 = \frac{1}{4\pi r_0\left|J_{in}\left(\frac{x+p}{2}\right)\right|^2} \qquad (26)$$

Next we turn to the evaluation of the terms containing β and β'. For all values of ν, the Bessel function $J_\nu(z)$ is defined by,

$$J_\nu(z) = \sum_{s=0}^{\infty}\frac{(-1)^s\,.\,(\frac{1}{2}z)^{\nu+2s}}{s!\,\Gamma(\nu+s+1)}$$

Hence,

$$\beta = J_{-in}(x) = \frac{(\frac{1}{2}x)^{-in}}{\Gamma(1+in)}\,.\left\{1 - \frac{(\frac{1}{2}x)^2}{1-in} + \ldots\right\}$$

For small values of x, we have,

$$\beta = \frac{(\frac{1}{2}x)^{-in}}{\Gamma(1-in)} \qquad \frac{1}{|\beta|^2} = |\Gamma(1-in)|^2 = \frac{n\pi}{\sinh n\pi} \qquad \ldots (27)$$

When order and argument both are large and comparable, a very good approximation to the Bessel function $J_\nu(z)$ is Watson (1922), page 229,

$$J_\nu(z) = M_\nu\cos\left(Q_\nu - \frac{\pi}{4}\right)$$

where

$$M_\nu = \left(\frac{2}{\pi\sqrt{z^2-\nu^2}}\right)^{\frac{1}{2}} \text{ and } Q_\nu = \sqrt{z^2-\nu^2} - \frac{\nu\pi}{2} + \nu\sin^{-1}\frac{\nu}{z}\,.$$

In the problem considered here $\nu = in$, hence

$$|\beta|^2 = |J_{-in}(x)|^2 = \frac{2}{\pi} \frac{\cosh^2(n\pi/2)}{\sqrt{x^2+n^2}}$$

or

$$|\beta|^2 = \frac{\sinh n\pi}{\pi\sqrt{x^2+n^2}} \qquad \dots \quad (28)$$

and also

$$|C|^2 = \frac{\{(\frac{x+p}{2})^2 + n^2\}^{\frac{1}{2}}}{4r.\sinh n\pi}$$

Therefore we have the following results,

$$\left.\begin{aligned} \left|\frac{C}{\beta}\right|^2 &= \frac{n^2\pi}{4r_0 \sinh^2 n\pi} \quad \text{for small } V_0 \\ \left|\frac{C}{\beta}\right|^2 &= \frac{\pi[(x^2+n^2)\{(\frac{x+p}{2})^2+n^2\}]^{\frac{1}{2}}}{4r_0\sinh^2 n\pi} \quad \text{for } x \text{ and } n \text{ large and comparable.} \\ \left|\frac{C}{\beta}\right|^2 &= \frac{\pi x(x+p)}{8r.\sinh^2 n\pi} \quad \text{for } x \text{ large and} >> n. \end{aligned}\right\} \dots \quad (29)$$

Making use of the relation

$$J'_\nu(z) = -\frac{\nu}{z} J_\nu(z) + J_{\nu-1}(z)$$

we have

$$\frac{\beta'}{\beta} = \frac{J'_{-in}(x)}{J_{-in}(x)} = \frac{ni}{x} + \frac{J_{-(in+1)}(x)}{J_{-in}(x)}$$

Since we have the relation

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} = \frac{\frac{1}{2}\frac{x}{\nu}}{1-\dfrac{\frac{1}{4}\frac{x^2}{\nu(\nu+1)}}{1-\dots\dots}} = \frac{1}{2}\frac{x}{\nu}\left\{1+\frac{x^2}{4\nu(\nu+1)}+\dots\right\}$$

therefore, for small values of x,

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} = \frac{1}{2}\frac{x}{\nu}$$

or

$$\frac{\beta'}{\beta} = \frac{ni}{x} - \frac{2ni}{x} = -\frac{ni}{x} = -ie^{\frac{1}{2}}.\left(\frac{E}{V_0}\right)^{\frac{1}{2}} \qquad \dots \quad (30)$$

For large n and for large values of x, we have

$$J_{-(in+1)}(x) \backsimeq J_{-in}(x).$$

Therefore

$$\frac{\beta'}{\beta} = 1 + \frac{ni}{x} = 1 + ie^{\frac{1}{2}}\sqrt{\frac{E}{V_0}} \qquad \dots \quad (31)$$

Using these relations, we get,

$$\left|\frac{\beta'}{\beta e^{\frac{1}{2}}}.\left(\frac{V_0}{V_1-E}\right)^{\frac{1}{2}} - 1\right|^2 = 1 + \frac{E}{V_1-E} = \frac{V_1}{V_1-E} \text{ for small values of } V_0. \qquad \dots \quad (32)$$

$$\left|\frac{\beta'}{\beta c^{\frac{1}{2}}}\left(\frac{V_0}{V_1-E}\right)^{\frac{1}{2}}-1\right|^2=\frac{1}{c(V_1-E)}\left\{V_0+cV_1-2\sqrt{cV_0(V_1-E)}\right\} \quad \dots \quad (33)$$

for V_0 comparable with E,

and $$=\frac{V_0}{c(V_1-E)} \text{ for } V_0 \text{ large.}$$

By the help of the relations (7), (23), (29) and (31), for very small values of V_0 we therefore have,

$$\lambda_0=\frac{4\sqrt{2}}{r_0\sqrt{M}}\cdot\frac{E}{V_1}\,(V_1-E)^{\frac{1}{2}}e^{-2P} \quad \dots \quad (34)$$

For very large values of V_0, we have, using the relations (6), (17), (23), (29) and (33),

$$\lambda_\infty=\frac{4(c^{\frac{1}{2}}+1)}{r_0\sqrt{2M}}\cdot(V_1-E)^{\frac{1}{2}}\cdot e^{-2P} \quad \dots \quad (35)$$

For values of V_0 comparable with E, we have, using (23), (29) and (33),

$$\lambda=\frac{4}{r_0}\sqrt{\frac{2}{M}}\cdot(V_1-E)^{\frac{1}{2}}\cdot\frac{(V_0+cE)^{\frac{1}{2}}\cdot\left\{\left(\frac{c^{\frac{1}{2}}+1}{2}\right)^2V_0+cE\right\}^{\frac{1}{2}}}{V_0+cV_1-2\sqrt{cV_0(V_1-E)}}\cdot e^{-2P} \quad \dots \quad (36)$$

It is to be noticed that λ_0 and λ_∞ are both independent of V_0 and that (36) reduces to (34) and (35) in the limiting case of V_0 very small and V_0 very large respectively. Also we have

$$\frac{\lambda_0}{\lambda_\infty}=\frac{2}{c^{\frac{1}{2}}+1}\cdot\frac{E}{V_1}=.76\,\frac{E}{V_1} \quad \dots \quad (37)$$

If we choose for r_0 the nuclear radius the value 9×10^{-13}cm. derived with one body model from experiment, Bethe (1937), for Z the average nuclear charge of radioactive elements, about 86, V_1 comes out to be 27 MV; and for E we choose an average energy 6 MV. Therefore the ratio λ_0/λ_∞ comes out to be .17 only. This leads us to conclude that the decay constant λ does not depend sensibly on the exact form and magnitude of the potential function inside the nucleus.

The most important factor is the exponential term and for the sake of completeness we discuss its nature. As defined in (17),

$$P=\frac{\sqrt{2M}}{h}\int_{r_0}^{r_1}\left(\frac{2Ze^2}{r}-E\right)^{\frac{1}{2}}dr.$$

Substituting $\cos^2\alpha=\frac{Er}{2Ze^2}$, we have;

$$P=\frac{\sqrt{2M}}{h}\cdot\frac{4Ze^2}{\sqrt{E}}\int_0^{\alpha_0}\sin^2\alpha\,d\alpha=\frac{\sqrt{2M}}{h}\cdot\frac{Ze^2}{\sqrt{E}}\left[2\alpha_0-\sin 2\alpha_0\right]$$

where
$$\theta_0 = \cos^{-1}\sqrt{\frac{E}{V_1}}\,.$$

Therefore
$$P = \frac{\sqrt{2M}}{h}\cdot\left\{\frac{2Ze^2}{\sqrt{E}}\right\}\left\{\cos^{-1}\sqrt{\frac{E}{V_1}} - \left(\frac{E}{V_1}\right)^{\frac{1}{2}}\left(1-\frac{E}{V_1}\right)^{\frac{1}{2}}\right\}$$

Defining
$$\frac{E}{V_1} = \theta$$

$$P = \frac{1.26Z}{\sqrt{E}}\,g(\theta) \qquad \ldots \quad (38)$$

where E is in million volts and

$$g(\theta) = \cos^{-1}\theta^{\frac{1}{2}} - \theta^{\frac{1}{2}}(1-\theta)^{\frac{1}{2}} \qquad \ldots \quad (39)$$

The values of $g(\theta)$ as a function of θ are given below :

TABLE I

θ	$g(\theta)$	θ	$g(\theta)$	θ	$g(\theta)$
·12	·8873	·18	·7485	·24	·6317
·13	·8651	·19	·7275	·25	·6142
·14	·8404	·20	·7071	·26	·5972
·15	·8161	·21	·6875	·27	·5822
·16	·7926	·22	·6683	·28	·5631
·17	·7702	·23	·6498	·29	·5483
				·30	·5328

For comparison with the experimental data, it is found convenient to use the experimental data for the decay constant and the energy of the alpha-paricle and to compute the nuclear radius from these data with the help of the derived theoretical formula. If the formula is correct, the radius r_0 must come out to be nearly the same for all radioactive nuclei. It has already been shown that the exact shape and magnitude of the potential function inside the nucleus plays only a very insignificant rôle in the problem and the ratio λ_r/λ_0 is of the order 5 (eq. 37), hence we adopt as the final formula,

$$\lambda = 2.5\lambda_0 = \frac{5\sqrt{2}}{e^2\sqrt{M}}\cdot\frac{E}{Z}\,(V_1 - E)^{\frac{1}{2}}e^{-2P} \qquad \ldots \quad (40)$$

This can be written in the form

$$\lambda = A(E, Z, r_0)e^{-2P}$$

or

$$\log_{10}\lambda = \log_{10}A - {\cdot}8686P$$

where $$\log_{10}A = 22.3822 + \log_{10}\frac{E}{Z} - \tfrac{1}{2}\log_{10}(V_1 - E)$$

E, V_1 being in million volts. In calculating A, we have taken the value of V_1 corresponding to $r_0 = 9.0 \times 10^{-13}$ cm. Thus we have, if V_1 be in million volts,

$$V_1 = .32Z.$$

Using the relation (38),

$$g(\theta) = \frac{\sqrt{E}}{1.995\,Z}(\log_{10}A - \log_{10}\lambda) \qquad \ldots \quad (41)$$

This enables us to calculate $g(\theta)$ from the experimental data of the decay constant λ and the energy E in million volts. The results given in Table I have been used to calculate θ and then the nuclear radius r_0 is given by the relation,

$$\log_{10}r_0 = 13.1501 + \log_{10}\lambda - \log_{10}\frac{E}{Z} \qquad \ldots \quad (42)$$

The calculated values of the nuclear radius r_0 are given in Table II and all radii except those of ThC, RaC lie between 7.8 and 9.7 × 10^{-13} cms. These are of the right order of magnitude and compare in a satisfactory manner with those obtained by Gamow (1929), (1937) and by Bethe (1937).

TABLE II

Disintegrating Nucleus	Product Nucleus		Energy E MV	Decay constant λ in sec^{-1}	Nuclear radius r_0 in 10^{-13} cm. calculated from (42)
Th	MTh1	88	4.34	$1.[illegible] \times 10^{-18}$	7.77
RdTh	ThX	88	5.52	1.15×10^{-8}	8.71
ThX	Thn	86	5.70	2.20×10^{-6}	[illegible].63
Thn	ThA	84	6.4[illegible]	1.27×10^{-2}	8.63
ThA	ThB	82	6.90	5.0	8.22
ThC	ThC''	81	6.20	6.7×10^{-5}	7.07
ThC'	ThD	[illegible]	8.95	†	—
UI	UXI	90	4.15	$5.[illegible] \times 10^{-18}$	9.67
Io	Ra	88	4.67	$2.9 \times 10^{-1[illegible]}$	9.25
Ra	Rn	86	4.88	$1.4[illegible] \times 10^{-11}$	8.65
Rn	RaA	84	5.50	2.10×10^{-6}	8.66
RaA	RaB	82	6.11	3.8×10^{-3}	8.48
RaC	RaC'	81	5.91	$[illegible] \times 10^{-7}$	7.21
RaC'	RaD	82	7.8[illegible]	7×10^{4}	9.07
RaF	RaG	82	5.[illegible]	5.7×10^{-8}	7.95
Pa	Ac	89	5.10	6.9×10^{-13}	7.95
RdAc	AcX	88	6.16	4.25×10^{-7}	7.81
AcX	An	86	5.82	7.2×10^{-7}	8.[illegible]
An	AcA	84	6.95	1.77×10^{1}	8.[illegible]
AcA	AcB	82	7.[illegible]	3.5×10^{2}	8.[illegible]5
AcC	AcC'	81	6.74	5.3×10^{-3}	6.80
AcC'	AcD	82	7.58	†	—

† No experimental values available

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